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**Submission date:** 02-Jan-2020 11:22AM (UTC+0700)

**Submission ID:** 1238974101

**File name:** Manuharawati_2019_J._Phys.__Conf._Ser._1417_012020.pdf (906.62K)

**Word count:** 2812

**Character count:** 12128

Journal of Physics: Conference Series

PAPER • OPEN ACCESS

# The New Convergence Definition for Sequence of *k*-Dimensional Subspaces of an Inner Product Space

To cite this article: Manuharawati *et al* 2019 *J. Phys.: Conf. Ser.* **1417** 012020

# The New Convergence Definition for Sequence of *k*-Dimensional Subspaces of an Inner Product Space

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**Abstract.** We discuss the convergence for sequence of subspaces of an inner product space. This paper is an extension of the work by Manuharawati et al [10 and 11]. In this paper, we present a concept of convergence of sequence of k-dimensional subspaces of an inner product space. The properties of the concept are established. Moreover, we also study its connection with angles in an inner product space.

Mathematics Subject Classification 2000: Primary 40A05; Secondary 14M15, 15A63, 46B20

## 1. Introduction

The concept of convergence of sequences of points has been extended by several authors (see [1–9]) to the concept of convergence of sequences of sets. The one of these such extensions that we will consider in this paper is Wijsman convergence. We will define convergence for sequences of subspaces and establish some basic results regarding these notions.

In [10,11], Manuharawati et al. have introduced a more geometrical definition of the convergence of a one-dimensional sequence of a one-dimensional subspace of a normed space and explained its connection with angle between two k-dimensional subspaces of a innerproduct space. Their definition of the convergence of a sequence of 1-dimensional subspace, however, is based on a definition convergence of a sequence on vector which we found a gap which we have generalized it to 2-dimensional subspace. The purpose of this note is to generalize their definition for k-dimensional subspaces and at the same time to explain the relationship between convergence of sequence with the angle between two k-dimensional subspaces of real inner product space.

Let us start with fundamental definitions from the literature. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space.

Distance between two vectors $u$ and $v$ in $X$.

$$d(u,v) = \|u - v\|$$

Distance between a vector $u$ and subspaces $V$ of $X$.

$$d(u,V) = \inf\{\|u - v\| : v \in V\} = \|u - u^V\|$$

Distance or gap between two subspaces of $X$ (see [12, pp. 197-199])

$$\begin{aligned} d(U,V) &= \sup\{\inf\{\|u-v\| : v \in V,\} : u \in U, \|u\| = 1\} \\ &= \sup\{\|u - \operatorname{proj}_V u\| : u \in U, \|u\| = 1\} \\ &= \sup_{\substack{u\in U \\ \|u\|=1}} \|u - \operatorname{proj}_V u\| \end{aligned}$$

Inspired from the distances, we can define the limit of a sequence of $k$-dimentional subspaces in the space.

In the following sections, an explicit definition for convergence of sequence of $k$-dimentional subspaces of a normed space will presented. As a consequence of our definition, the continuity property of angle between two $k$-dimensional subspaces of inner product spaces is obtained.

## 2. Results

Given an inner product space $(X, \langle\cdot,\cdot\rangle)$ of dimension $k$ or higher (may be infinite). Let $U_n$ and $V$ be $k$-dimensional subspaces of $X$ for every $n \in \mathbb{N}$.

**Definition 2.1**. We shall say that the sequence $(U_n)$ converges to the subspace $V$, written $U_n \to V$ if

$$d(U_n, V) = \sup_{\substack{u_n\in U \\ \|u_n\|=1}} \|u_n - \operatorname{proj}_V u_n\| \to 0$$

as $n \to \infty$.

Let $U_n = span\{u_{1_n}, u_{2_n}, \dots, u_{k_n}\}$ and $V = span\{v_1, v_2, \dots, v_k\}$ be $k$-dimensional subspaces of $X$, where $\{u_{1_n}, u_{2_n}, \dots, u_{k_n}\}$ and $\{v_1, \dots, v_k\}$ are orthonormal for every $n \in \mathbb{N}$. The following theorems are some of the properties of the convergence concept for sequence of $k$ –dimensional subspace of $X$.

**Theorem 2.2.** *$U_n \to V$ if and only if $\|u_{i_n} - \operatorname{proj}_V u_{i_n}\| \to 0$ for every $i = 1,2,\dots,k$.*

***Proof.***
Given $U_n \to V$, then by Definition 2.1, $\sup_{\substack{u_n\in U \\ \|u_n\|=1}} \|u_n - \operatorname{proj}_V u_n\| \to 0$. It follows that for every $i = 1,2,\dots,k$ $u_{i_n} \in U_n$ with $\|u_{i_n}\| = 1$ then

$$\|u_{i_n} - \operatorname{proj}_V u_{i_n}\| \le \sup_{\substack{u_n\in U_n \\ \|u_n\|=1}} \|u_n - \operatorname{proj}_V u_n\| \to 0.$$

Conversely, we know that $\operatorname{proj}_V u_{i_n} = \sum_{j=1}^{k} \langle u_{i_n}, v_j\rangle v_j$ and for every $i = 1,2,\dots,k$

$$\begin{aligned} &\sup_{\substack{u_n\in U_n \\ \|u_n\|=1}} \|u_n - \operatorname{proj}_V u_n\| \\ &= \sup_{\substack{u_n\in U_n \\ \|u_n\|=1}} \left\|\left(\alpha_1 u_{1_n} + \alpha_2 u_{2_n} + \dots + \alpha_k u_{k_n}\right) - \operatorname{proj}_V \left(\alpha_1 u_{1_n} + \alpha_2 u_{2_n} + \dots + \alpha_k u_{k_n}\right)\right\| \\ &\le \sup_{\substack{u_n\in U_n \\ \|u_n\|=1}} \sum_{i=1}^{k} \left\|\alpha_i u_{i_n} - \operatorname{proj}_V\left(\alpha_i u_{i_n}\right)\right\| \\ &= \sum_{i=1}^{k} |\alpha_i| \left\|u_{i_n} - \operatorname{proj}_V\left(u_{i_n}\right)\right\| \to 0. \end{aligned}$$

■

**Theorem 2.3.** *$U_n \to V$ if and only If $\sum_{j=1}^{k} \langle u_{i_n}, v_j\rangle^2 \to 1$ for every $i = 1,2,\dots,k$.*

***Proof.***

We know that

$$\operatorname{proj}_V u_{i_n} = \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle v_j.$$

Since $\{v_1, v_2, \dots, v_k\}$ is orthonormal, we can get

$$\begin{aligned}
\left\| u_{i_n} - \operatorname{proj}_V u_{i_n} \right\|^2 &= \langle u_{i_n} - \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle v_j \,, u_{i_n} - \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle v_j \rangle \\
&= \langle u_{i_n}, u_{i_n} \rangle - 2 \langle u_{i_n}, \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle v_j \rangle + \langle \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle v_j \,, \sum_{l=1}^{k} \langle u_{i_n}, v_l \rangle v_l \rangle \\
&= 1 - 2 \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle \langle u_{i_n}, v_j \rangle + \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle \left( \sum_{l=1}^{k} \langle u_{i_n}, v_l \rangle \langle v_j, v_l \rangle \right) \\
&= 1 - \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2.
\end{aligned}$$

So, $\left\| u_{i_n} - \operatorname{proj}_V u_{i_n} \right\| \to 0$ if and only if $\sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2 \to 1$ as $n \to \infty$. By Theorem 2.2, we get $U_n \to V$ if and only if $\sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2 \to 1$ as $n \to \infty$.

■

**Theorem 2.4.** $U_n \to V$ *if and only if* $\left\| \operatorname{proj}_V u_{i_n} \right\| \to 1$ *for every* $i = 1,2, \dots, k$.

***Proof.***
Since $\{v_1, v_2, \dots, v_k\}$ is orthonormal, we can get

$$\begin{aligned}
\left\| \operatorname{proj}_V u_{i_n} \right\|^2 &= \langle \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle v_j \,, \sum_{l=1}^{k} \langle u_{i_n}, v_l \rangle v_l \rangle \\
&= \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle \left( \sum_{l=1}^{k} \langle u_{i_n}, v_l \rangle \langle v_j, v_l \rangle \right) \\
&= \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2
\end{aligned}$$

So, $\sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2 \to 1$ if and only if $\left\| \operatorname{proj}_V u_{i_n} \right\| \to 1$ as $n \to \infty$. By Theorem 2.2, we get $U_n \to V$ if and only if $\left\| \operatorname{proj}_V u_{i_n} \right\|$ as $n \to \infty$.

■

Hereafter we shall employ the standard $n$-norm $\|\cdot,\cdot, \dots, \cdot\|$ on X, given by

$$\|u_1, u_2, \dots, u_n\| := \begin{vmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle & \cdots & \langle u_1, u_n \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle & \cdots & \langle u_2, u_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_1, u_n \rangle & \langle u_2, u_3 \rangle & \cdots & \langle u_n, u_n \rangle \end{vmatrix}^{\frac{1}{2}}$$

Here assume $n > 2$ (see {6} and {10}). Note particularly that $\|u_1, u_2, \dots, u_n\| = \sqrt{\det[\langle u_i, u_j \rangle]}$ is nothing but the Gram's determinant generated by $u_1, u_2, \dots u_n$ (see [13] or [14]). Geometrically, being the square root of the Gram's determinant, $\|u_1, u_2, \dots, u_n\|$ represents the volume of the $n$-dimensional parallelepiped spanned by $u_1, u_2, \dots, u_n$.

The following theorems are properties of convergent sequence related standard n-norm.

**Theorem 2.5**. $U_n \to V$ *if and only if* $\left\| u_{i_n}, \mathrm{proj}_V\, u_{i_n} \right\| \to 0$ *as* $n \to \infty$ *for every* $i = 1,2,\dots,k$.
***Proof.***
We know that

$$\mathrm{proj}_V\, u_{i_n} = \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle v_j.$$

Since $\{v_1, v_2, \dots, v_k\}$ is orthonormal, we can observe that

$$\begin{aligned}
\left\| u_{i_n}, \mathrm{proj}_V\, u_{i_n} \right\|^2 &= \begin{vmatrix} \langle u_{i_n}, u_{i_n} \rangle & \langle u_{i_n}, \mathrm{proj}_V\, u_{i_n} \rangle \\ \langle \mathrm{proj}_V\, u_{i_n}, u_{i_n} \rangle & \langle \mathrm{proj}_V\, u_{i_n}, \mathrm{proj}_V\, u_{i_n} \rangle \end{vmatrix} \\
&= \langle u_{i_n}, u_{i_n} \rangle \langle \mathrm{proj}_V\, u_{i_n}, \mathrm{proj}_V\, u_{i_n} \rangle - \langle u_{i_n}, \mathrm{proj}_V\, u_{i_n} \rangle^2 \\
&= \langle u_{i_n}, u_{i_n} \rangle \left\langle \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle v_j, \sum_{l=1}^{k} \langle u_{i_n}, v_l \rangle v_l \right\rangle - \left( \left\langle u_{i_n}, \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle v_j \right\rangle \right)^2 \\
&= \langle u_{i_n}, u_{i_n} \rangle \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle \left\langle v_j, \sum_{l=1}^{k} \langle u_{i_n}, v_l \rangle v_l \right\rangle - \left( \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2 \right)^2 \\
&= \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2 - \left( \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2 \right)^2 \\
&= \left( 1 - \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2 \right) \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2.
\end{aligned}$$

As for inner product, we have Cauchy-Schwarz inequality, then $\left| \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2 \right| \le \sum_{j=1}^{k} \left\| u_{i_n} \right\|^2 \left\| v_j \right\|^2 = k$. Consequently, $\left\| u_{i_n}, \mathrm{proj}_V\, u_{i_n} \right\| \to 0$ if and only if $\sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2 \to 1$ as $n \to \infty$. By Theorem 2.3, we get $U_n \to V$ if and only if $\left\| u_{i_n}, \mathrm{proj}_V\, u_{i_n} \right\| \to 0$ as $n \to \infty$.

■

**Theorem 2.6**. $U_n \to V$ *if and only if* $\left\| u_{i_n}, v_1, v_2, \dots, v_n \right\| \to 0$ *as* $n \to \infty$ *for every* $i = 1,2,\dots,k$.
***Proof.***
Since $\{v_1, v_2, \dots, v_k\}$ is orthonormal, we can observe that

$$\begin{aligned}
\left\| u_{i_n}, v_1, v_2, \dots, v_k \right\|^2 &= \begin{vmatrix} \langle u_{i_n}, u_{i_n} \rangle & \langle u_{i_n}, v_1 \rangle & \langle u_{i_n}, v_2 \rangle & \dots & \langle u_{i_n}, v_k \rangle \\ \langle v_1, u_{i_n} \rangle & \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_k \rangle \\ \langle v_2, u_{i_n} \rangle & \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \dots & \langle v_2, v_k \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle v_k, u_{i_n} \rangle & \langle v_k, v_1 \rangle & \langle v_k, v_2 \rangle & \dots & \langle v_k, v_k \rangle \end{vmatrix} \\
&= \begin{vmatrix} 1 & \langle u_{i_n}, v_1 \rangle & \langle u_{i_n}, v_2 \rangle & \dots & \langle u_{i_n}, v_k \rangle \\ \langle v_1, u_{i_n} \rangle & 1 & 0 & \dots & 0 \\ \langle u_{i_n}, v_2 \rangle & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle v_k, u_{i_n} \rangle & 0 & 0 & \dots & 1 \end{vmatrix} \\
&= 1 - \langle u_{i_n}, v_1 \rangle^2 + \langle u_{i_n}, v_2 \rangle^2 + \dots + \langle u_{i_n}, v_k \rangle^2
\end{aligned}$$

$$= 1 - \sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2$$

So, $\sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2 \to 1$ if and only if $\|u_{i_n}, v_1, v_2, \dots, v_k\| \to 0$ as $n \to \infty$. By Theorem 2.3, we get $U_n \to V$ if and only if $\|u_{i_n}, v_1, v_2, \dots, v_k\| \to 0$ as $n \to \infty$.

■

Remark 2.7. Theorem 2.2 until Theorem 2.6 said that the following statements are equivalent:

(i) $U_n \to V$

(ii) $\|u_{i_n} - \mathrm{proj}_V\, u_{i_n}\| \to 0$ *for every* $i = 1,2,\dots,k$.

(iii) $\sum_{j=1}^{k} \langle u_{i_n}, v_j \rangle^2 \to 1$ *for every* $i = 1,2,\dots,k$.

(iv) $\|\mathrm{proj}_V\, u_{i_n}\| \to 1$ *for every* $i = 1,2,\dots,k$.

(v) $\|u_{i_n}, \mathrm{proj}_V\, u_{i_n}\| \to 0$ *as* $n \to \infty$ *for every* $i = 1,2,\dots,k$.

(vi) $\|u_{i_n}, v_1, v_2, \dots, v_n\| \to 0$ *as* $n \to \infty$ *for every* $i = 1,2,\dots,k$.

**Theorem 2.8. (Homogeneity Property)** *The Definition 2.1 satisfy homogeneity property. It means, the definition is dependent of the choice of orthonormal bases for V and* $U_n$ *for every* $n \in \mathbb{N}$.

***Proof.***

First note that the statement of

$$\|u_{i_n}, v_1, v_2, \dots, v_n\| \to 0, \forall i = 1,2$$

is independent of the choice of basis for $U_n$ and $V$ for every $n \in \mathbb{N}$. Further, since the value is also invariant under any change of basis for $U_n$ and $V$. Indeed, the value is unchanged if we

(a) swap $u_{1_n}$ and $u_{2_n}$,

(b) swap $v_1$and $v_2$,

(c) replace $u_{1_n}$ by $\frac{\alpha u_{1_n} + \beta u_{2_n}}{\|\alpha u_{1_n} + \beta u_{2_n}\|}$,

(d) replace $v_1$ by $\frac{\alpha v_1 + \beta v_2}{\|\alpha v_1 + \beta v_2\|}$, with $\alpha \neq 0$ or $\beta \neq 0$.

■

**Corollary 2.9.** $U_n \to V$ if and only if $\sum_{l=1}^{k} \langle u_{i_n} + u_{j_n}, v_l \rangle^2 \to 2$ *for every* $i = 1,2,\dots,k$.

***Proof.***

Let $U_n$ be k-dimensional subspace with an orthonormal basis $u_{1_n}, u_{2_n}, \dots, u_{k_n}$. Using Gram-Schmidt orthonormal process, we can construct another orthonormal basis, say $u'_{1_n}, u'_{2_n}, \dots, u'_{k_n}$, with $u'_1 = \frac{u_{i_n} + u_{j_n}}{\sqrt{2}}$ for sum $i,j \in \{1,2,\dots,k\}$. According to Homogeneity Property and Theorem 2.3, we get

$$0 \leftarrow \sum_{l=1}^{k} \langle \frac{u_{i_n} + u_{j_n}}{\sqrt{2}}, v_l \rangle^2 = \sum_{l=1}^{k} \frac{1}{2} \langle u_{i_n} + u_{j_n}, v_l \rangle^2 = \frac{1}{2} \sum_{l=1}^{k} \langle u_{i_n} + u_{j_n}, v_l \rangle^2$$

as $n \to \infty$.

■

**Corollary 2.10.** $U_n \to V$ if and only if $\sum_{l=1}^{k} \langle u_{i_n}, v_l \rangle \langle u_{j_n}, v_l \rangle \to 0$ *where* $i \neq j$ *for every* $i,j = 1,2,\dots,k$.

***Proof.***

Observe that

$$\langle u_{i_n} + u_{j_n}, v_l \rangle^2 = \left( \langle u_{i_n}, v_l \rangle + \langle u_{j_n}, v_l \rangle \right)^2 = \langle u_{i_n}, v_l \rangle^2 + \langle u_{j_n}, v_l \rangle^2 + 2 \langle u_{i_n}, v_l \rangle \langle u_{j_n}, v_l \rangle.$$

Thus, we have

$$\langle u_{i_n}, v_l \rangle \langle u_{j_n}, v_l \rangle = \frac{1}{2} \left( \langle u_{i_n} + u_{j_n}, v_l \rangle^2 - \langle u_{i_n}, v_l \rangle^2 - \langle u_{j_n}, v_l \rangle^2 \right).$$

Consequently, if $i \neq j$ then

$$\sum_{l=1}^{k}\langle u_{i_n}, v_l\rangle\langle u_{j_n}, v_l\rangle = \frac{1}{2}\left(\sum_{l=1}^{k}\langle u_{i_n}+u_{j_n}, v_l\rangle^2 - \sum_{l=1}^{k}\langle u_{i_n}, v_l\rangle^2 - \sum_{l=1}^{k}\langle u_{j_n}, v_l\rangle^2\right).$$

Using Theorem 2.3 and Corollary 2.9, we get $U_n \to V$ if and only if $\sum_{l=1}^{k}\langle u_{i_n}, v_l\rangle\langle u_{j_n}, v_l\rangle \to 0$ as $n \to \infty$.

■

The following theorem stated that the limit of sequences of k-dimensional subspaces is unique.

**Theorem 2.11. (Uniqueness Property)** *If* $U_n \to V$ *and* $U_n \to W$ *as* $n \to \infty$, *then* $V = W$.

***Proof.***
Let $U_n \to V$ as $n \to \infty$. By Definition 3.1, we have

$$\left\|u_{i_n}, v_1, v_2, \dots, v_k\right\| \to 0, \forall i = 1,2$$

as $n \to \infty$.
Since $U_n \to W$ as $n \to \infty$ then we also have

$$\left\|u_{i_n}, w_1, w_2, \dots, w_k\right\| \to 0, \forall i = 1,2$$

as $n \to \infty$.
By using Triangle Inequality, we get

$$\begin{aligned}\|w_i, v_i, v_2, \dots, v_k\| &= \left\|w_i - u_{i_n} + u_{i_n}, v_1, v_2, \dots, v_k\right\| \\ &\le \left\|w_1 - u_{1_n}, v_1, v_2, \dots, v_k\right\| + \left\|u_{1_n}, v_1, v_2, \dots, v_k\right\|.\end{aligned}$$

■

Example 2.12. *If* $U_n = span\{u_1, u_2, \dots, u_n\} = U$ *then* $U_n \to U$ *as* $n \to \infty$. For every $i = 1,2$

$$\left\|u_{i_n}, u_1, u_2 \dots, u_n\right\| = 0 \to 0.$$

## 3. Furthermore Results

Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. $U = span\{u1, u_2, \dots, u_k\}$ and $V = span\{v_1, v_2, v_3, \dots, v_k\}$ are $k$-dimensional subspaces of $X$, then the angle between subspaces $U$ and V is defined by $\theta(U,V)$ with $0 \le \theta(U,V) < \frac{\pi}{2}$ and $\cos\theta(U,V) = \|\mathrm{proj}_V\, u_1, \mathrm{proj}_V\, u_2, \dots, \mathrm{proj}_V\, u_k\|$ (see [15,16,17]).

**Theorem 3.3.** Let $U_n = span\{u_{1_n}, u_{2_n}, \dots, u_{k_n}\}$ and $V = span\{v_1, v_2, \dots, v_k\}$ be $k$-dimensional subspaces of $X$, where $\{u_{1_n}, u_{2_n}, \dots, u_{k_n}\}$ and $\{v_1, \dots, v_k\}$ are orthonormal for every $n \in \mathbb{N}$. $U_n \to V$ *if and only if* $\theta(U_{i_n}, V) \to 0$ *as* $n \to \infty$.
***Proof.***
Let

$$\begin{aligned}\|\mathrm{proj}_V\, u_1, \mathrm{proj}_V\, u_2, \dots, \mathrm{proj}_V\, u_k\|^2 &= \det(\langle \mathrm{proj}_V\, u_i, \mathrm{proj}_V\, u_j\rangle) \\ &= \det\left(\langle\sum_{l=1}^{k}\langle u_{i_n}, v_l\rangle v_l\,, \sum_{m=1}^{k}\langle u_{j_n}, v_m\rangle v_m\rangle\right) \\ &= \det\left(\sum_{l=1}^{k}\langle u_{i_n}, v_l\rangle\langle v_l, \sum_{m=1}^{k}\langle u_{j_n}, v_m\rangle v_m\rangle\right)\end{aligned}$$

$$= \det\left(\sum_{l=1}^{k} \langle u_{i_n}, v_l \rangle \left( \sum_{m=1}^{k} \langle u_{j_n}, v_m \rangle \langle u_j, v_m \rangle \right)\right)$$
$$= \det\left(\sum_{l=1}^{k} \langle u_{i_n}, v_l \rangle \langle u_{j_n}, v_m \rangle \right)$$

According to Corollary 2.10, If $U_n \to V$, then $\sum_{l=1}^{k} \langle u_{i_n}, v_l \rangle \langle u_{j_n}, v_m \rangle \to 0$ where $i \neq j$ for every $i, j = 1,2, \dots, k$. Because the determinant function is continuous, then we have
$$\|\mathrm{proj}_V\, u_1, \mathrm{proj}_V\, u_2, \dots, \mathrm{proj}_V\, u_k\|^2 \to 1.$$
Then the arccosinus function is continuous, the we get
$$\theta\left(U_{i_n}, V\right) \to \arccos 1 = 0$$
as $n \to \infty$.

■

## Acknowledgment

I would like to thank DIPA State University of Surabaya for financial support.

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